Homotopical characterization of strong contextuality

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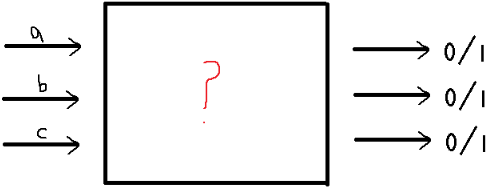
joint with Cihan Okay.

Quantum contextuality

Experiments in quantum physics:



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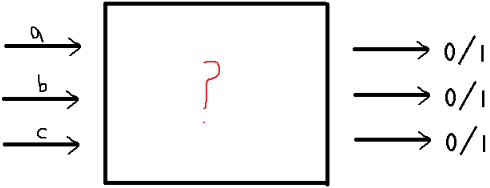

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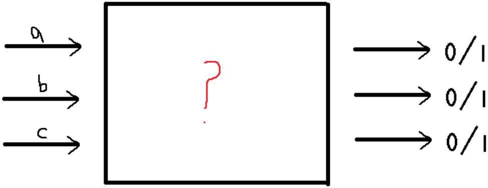

- Every time we can measure only two of them.
- We can do the three experiments (together) as much as we want.

Quantum contextuality(cont.)

So we get the following probability tables:

(a,b)		
	p^{00}	p^{01}
	p^{10}	p^{11}

,

(b,c)		
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We always have

$$p^{00}+p^{01}=s^{00}+s^{01}\ ,\ \ p^{00}+p^{10}=q^{00}+q^{01}\ ,\ \ q^{00}+q^{10}=s^{00}+s^{10}$$

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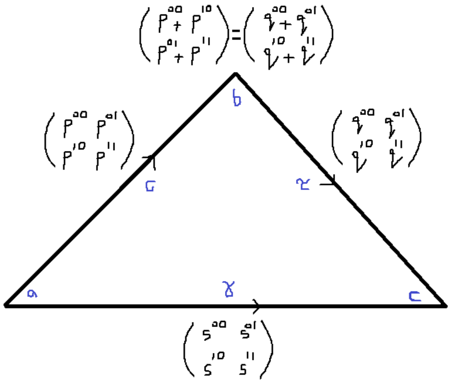
In this case the tables called *contextual* tables.

Topological description



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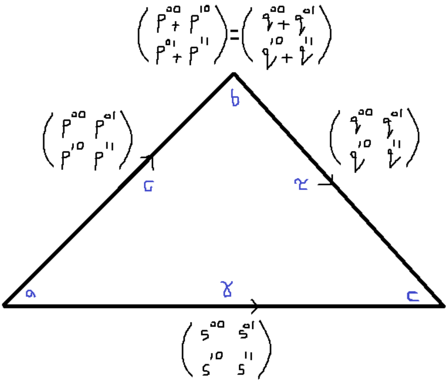

Or using commutative diagrams:

$$
\begin{array}{ccc}
\{\sigma, \tau, \gamma\} & \xrightarrow{p_1} & D(\mathbb{Z}_2 \times \mathbb{Z}_2) \\
d_0 \downarrow\downarrow d_1 & & D(d_0) \downarrow\downarrow D(d_1) \\
\{a, b, c\} & \xrightarrow{p_0} & D(\mathbb{Z}_2)
\end{array}
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Simplicial distributions

A *simplicial set* X is a collection of sets $X_0, X_1, X_2, \ldots$ with a face and degeneracy maps. In X_n we have the n-simplices.

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Example

The simplicial set $\Delta_{\mathbb{Z}_n}$

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Definition: A *simplicial distribution* is a simplicial map

$$p : X \to D(\Delta_{\mathbb{Z}_n})$$

X is the measurement space and $\Delta_{\mathbb{Z}_n}$ is the outcome space.

Strong contextuality

Given a measurement space X. The set of all the simplicial distributions $p : X \to D(\Delta_{\mathbb{Z}_n})$ form a polytope.

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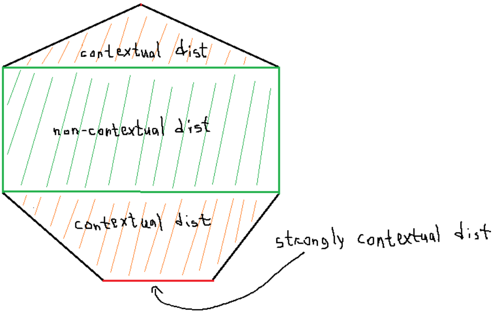

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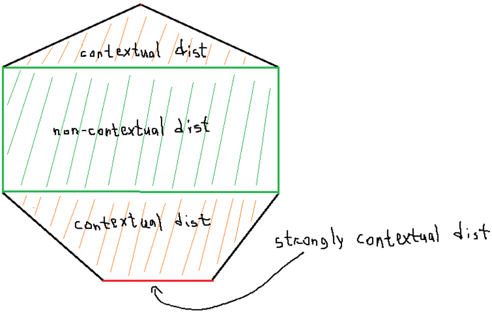

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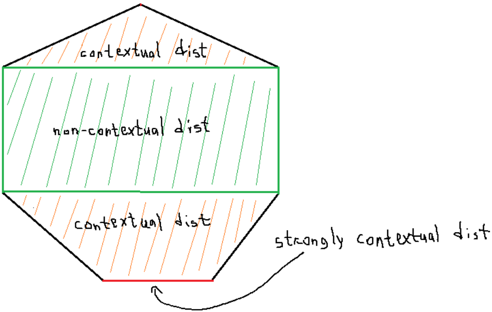

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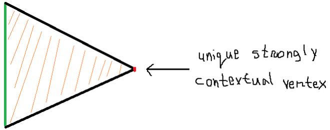

$\Delta_{\mathbb{Z}_n}$ as a path space of the nerve of $\mathbb{Z}_n$

Fact: As a topological space, $\Delta_{\mathbb{Z}_n}$ is the set of paths in $N\mathbb{Z}_n$ that start at some fixed point. We have a map $\kappa : \Delta_{\mathbb{Z}_n} \to N\mathbb{Z}_n$, send the path to it's terminal point.

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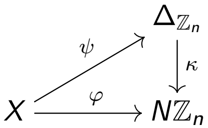

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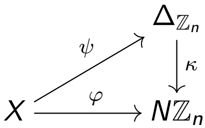

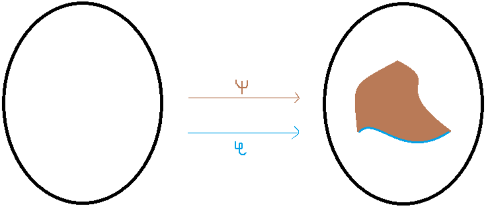

Detecting strong contextuality using homotopy

Proposition:

Given a simplicial distribution $p : X \to D(\Delta_{\mathbb{Z}_n})$. If there is a subspace $Z \subseteq X$ and a simplicial map $\varphi : Z \to N\mathbb{Z}_n$ which is not null-homotopic, such that

$$
\begin{array}{ccccc}
Z & \hookrightarrow & X & \xrightarrow{\;p\;} & D(\Delta_{\mathbb{Z}_n}) \\
 & \searrow^{\varphi} & & & \downarrow D(\kappa) \\
 & & N\mathbb{Z}_n & \xhookrightarrow[\delta_{N\mathbb{Z}_n}]{} & D(N\mathbb{Z}_n)
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$$

then $p : X \to D(\Delta_{\mathbb{Z}_n})$ is strongly contextual.

$D(\Delta_{\mathbb{Z}_n})$ is a compository

A *compository* (Compositories and Gleaves by C.Flori and T.Fritz) is a simplicial set equipped with a composition operation:

m-simplex and n-simplex which have a common k-simplex face turns into an $(m+n-k)$-simplex.

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Given a small category $\mathbf{C}$. The nerve $N\mathbf{C}$ is a "trivial" compository.

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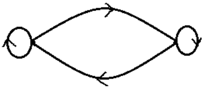
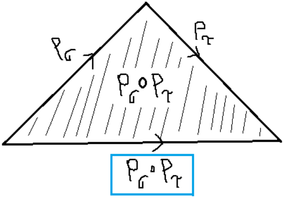


Simplicial distribution as a category

The composition that we need:



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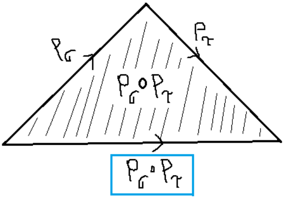

If the measurement space X is 1-skeletal (directed graph), we can think about a simplicial distribution $p : X \to D(\Delta_{\mathbb{Z}_n})$ as a category. We denote this category by $\mathbf{C}(X, p)$.

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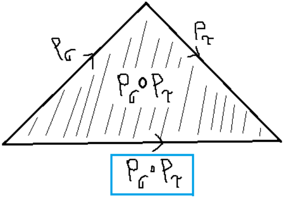

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Proposition

A simplicial distribution $p : X \to D(\Delta_{\mathbb{Z}_2})$ is strongly contextual if and only if there is $a \in X_0$ and $A \in \mathbf{C}(X, p)(a, a)$ such that A is the unique strongly contextual as a simplicial distribution on the one edge circle.

The homotopical characterization

Using the Proposition above, we get the following result:

Theorem

Let X be a 1-skeletal measurement space. A simplicial distribution $p : X \to D(\Delta_{\mathbb{Z}_2})$ is strongly contextual if and only if there is a circle $C \subseteq X$ and a simplicial map $\varphi : C \to N\mathbb{Z}_2$ which is not null-homotopic, such that

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Thank you for listening